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The affine algebraic curve of Newton maps in Przytycki's cubic rational maps

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4.1 Introduction

Przytycki studies a family of cubic rational maps:

$$\mathcal{PC}(a,b,c)=\{f(z)=z^2+c+\frac{b}{z-a}\ :a,b,c\in\mathbf{C}\},$$

and defines the exotic map to be a map of $\mathcal{PC}(a,b,c)$ having two super-attracting fixed points and a critical point of period 2. An example of exotic map is given in Przytycki [Prz94], obtained by computer experiment.

Example: $f(z)=z^2+c+b/(z-a):c=-3.121092, a=1.719727, b=0.3142117.$

He considers 1-parameter families joining exotic examples with Newton maps for degree 3 polynomials. The subfamily with a super-attracting fixed point except of ∞ can be parametrized by two parameters (k,w) as follows:
let w be super-attracting fixed point, and $a=kw$. Then

$$a=kw,\ b=2w^3(1-k)^2,\ c=w^2(2k-3)+w,$$

and

$$\mathcal{PC}(w,k)=\left\{f(z)=z^2+w^2(2k-3)+w+\frac{2w^3(1-k)^2}{z-kw}\right\}.$$

Four critical points are ∞, w and

$$u=\frac{w(-1+2k-\sqrt{4k-3})}{2},\quad v=\frac{w(-1+2k+\sqrt{4k-3})}{2}$$

.

Przytycki gives two questions in [Prz94]:

Question 1 In the set of Newton maps for the polynomials $P_\lambda = z^3+(\lambda-1)z-\lambda$, there exist Mandelbrot-like sets where the free critical point converges to a periodic attrcting orbit. For a critical point $v = g(k,w)$, what happens to $\mathcal{M}(v)$-sections of the set of exotic maps when we change parameters from Newton maps to the exotic ones ? : namely, do these sets move to $\mathcal{M}(v)$ (or $\mathcal{M}(u)$)-sections of the set of exotic maps when we change parameters from Newton maps to the exotic ones ?

Question 2 Describe precisely how does the dynamics bifurcate for real parameters k, w.

Concerning about these questions, we obtain following results: to the first question, there is an affine algebraic curve consisting of Newton maps for degree 3 polynomials, and to the second one, we partly give an answer. Namely,for any fixed parameter k, we consider the bifurcation of this family as the parameter w varies monotonely. We observe complex bifurcations for $3/4 < k < 1$.

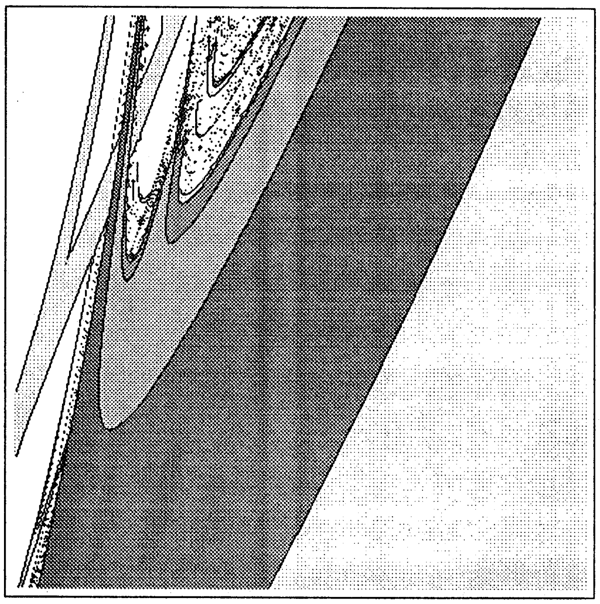

図 1 Bifurcations in real (k,w)-plane

4.2 Cubic Newton Curve

We show that there is an affine algebraic curve consisted by all Newton maps in $\mathcal{PC}(a,b,c)$. We call hereafter this curve **cubic Newton curve**, denoted by $\mathcal{N}(w,k)$.

Proposition 1 *The defining equation of the cubic Newton curve is the follwing:*

$$\mathcal{N}(w,k): \ k^2w^2 - 6kw^2 + 9w^2 - 2kw - 3w + 1 = 0.$$

The cubic Newton curve is irreducible of genus one with two singular points

$$(0,0,1), \ \ (0,1,0)$$

on $L_\infty : Z = 0$ *of* $\mathbf{P}^2\mathbf{C} : (Z,k,w)$.

Outline of proof:

$$f(k,w,x) = x^2 + w^2\,(2k-3) + w + \frac{2\,w^3\,(1-k)^2}{(x-kw)},$$

$$f'(k,w,x) = 2\,x - \frac{2\,(1-k)^2\,w^3}{(x-k\,w)^2}.$$

The critical points are ∞ , w, and

$$u = \frac{-w\sqrt{4k-3} - (1-2k)\,w}{2}, \quad v = \frac{w\sqrt{4k-3} - (1-2k)\,w}{2}.$$

For f to be a Newton map, we claim $f(k,w,u) = u$, or $f(k,w,v) = v$.

Therefore the equation of the Newton curve is

$$(k^2w^2 - 6kw^2 + 9w^2 - 2kw - 3w + 1) = 0.$$

Let

$$F(Z,k,w) = Z^4 - 3wZ^3 - 2z^2kw + 9Z^2w^2 - 6Zkw^2 + k^2w^2 = 0,$$

and P_k : $(0,0,1)$, P_w : $(0,1,0)$. The singular points are P_k, P_w. The principal part of F is $(3Z-k)^2$. Therefore by Plücker's formula, we can calculate the genus one.

We define dynamical curves in the parameter space:

Definition Let $\mathrm{Per}_p(\mu)$ consist of all parameter pairs (k,w) for which the associated cubic rational map $f(k,w,x)$ has a periodic orbit of period p with multiplier $(f^p)'$ equal to μ.

In particular, $\mathrm{Per}_1(0)$ consists of all parameter pairs with a super-attracting fixed point. The real part of the curve $\mathrm{Per}_1(1)$ consists of all parameter pairs for which the graph of f is tangent to the diagonal. Such points of tangency are called **saddle nodes** of period 1. On $\mathrm{Per}_1(-1)$ attracting period one orbits bifurcate into attracting period two orbits.

It is clear that (k, w) of $\mathrm{Per}_1(-1)$ belongs to $\mathrm{Per}_2(1)$, but it is not known whether inverse inclusion holds. In the quadratc rational maps, we have $\mathrm{Per}_2(1) = \mathrm{Per}_1(-1)$.

Proposition 2 *We obtain defining equations of two dynamical curves:*

$Per_1(1):\ k^2w^2 - 6kw^2 + 9w^2 - 2kw - 2w + 1 = 0,$

$Per_1(-1):\ 3k^2w^2 - 18kw^2 + 27w^2 - 6kw - 14w + 3 = 0.$

See Figures for plots of these curves in the real case.

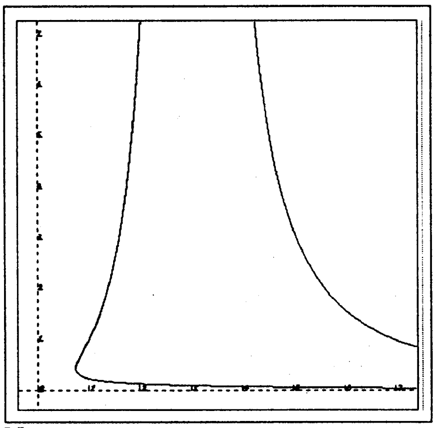

図 2 Newton curve :$(k^2 - 6k + 9)w^2 - (2k + 3)w + 1 = 0.$

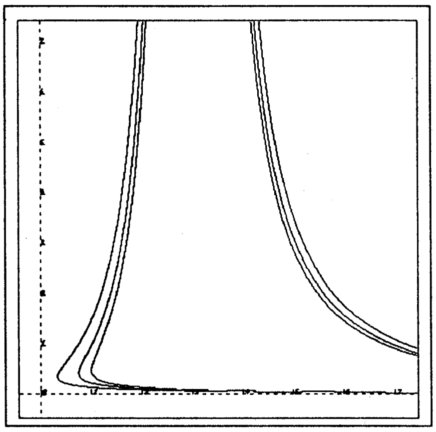

図 3 Dynamical curves :$Per_1(1)$, $Per_1(-1)$.

Outline of proof:

The fixed points of $f(k, w, x)$ are ∞, w and

$$x_1 = \frac{(1-k)\,w - 1 - \sqrt{(k^2 - 6k + 9)w^2 + (-2k-2)w + 1}}{2},$$

$$x_2 = \frac{-(1-k)\,w + 1 + \sqrt{(k^2 - 6k + 9)w^2 + (-2k-2)w + 1}}{2}.$$

From $f'(k, w, x_i) = 1$, (resp. $= -1$) for $i = 1$, or 2, we obtain the equation of $Per_1(1)$ (resp. $Per_1(-1)$).

4.3 bifurcations

Proposition 3 *For the real parameter k, we can rougly divide (k, w)-plane into four distinct classes as follows: (1) $k < 3/4$, (2) $3/4 \le k < 1$, (3) $1 < k < 3/2$, (4) $3/2 \le k$.*

Outline of Proof:

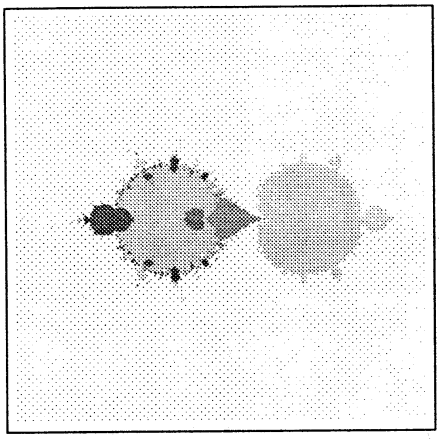

図4 **Bifurcation:**$k = 1.15$, $w : (-1.5, -2) - (2.5, 2)$

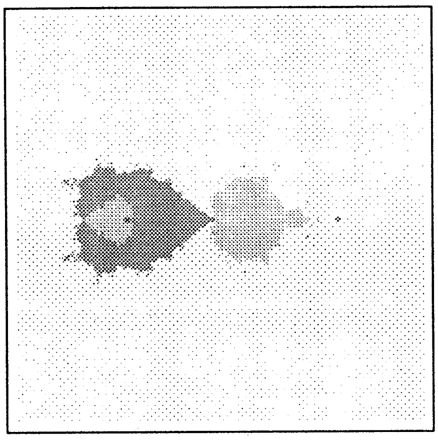

図5 **Bifurcation:** $k = 1.5$, $w : (-2, -4) - (6, 4)$

• For k : $3/4 \leq k \leq 1$, the critical points u, v, w except ∞ are ordered as $u < v < a < w$.

• For k : $1 \leq k < 3/2$, $u < w < a < v$.

• For k : $3/2 \leq k$, $w < u < a < v$. in case (3) (resp.(4)), the dynamics of u (resp. v)is analyzed as the dynamics of a quadratic map by Douaddy-Hubbard-Sullivan's theory([D-H85]). In case (2), the dynamical behavior is very complex. In case (1), two critical points u, v are compex numbers, mutually complex conjugate. The dynamical behavior is very complex: namley we can observe "swallow" and "tri-corn" configulations ([Mil90]).

参考文献

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